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Global Anomalies in Chiral Gauge Theories on the Lattice

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Abstract

We discuss the issue of global anomalies in chiral gauge theories on the lattice. In Lüscher's approach, these obstructions make it impossible to define consistently a fermionic measure for the path integral. We show that an $SU(2)$ theory has such a global anomaly if the Weyl fermion is in the fundamental representation. The anomaly in higher representations is also discussed. We finally show that this obstruction is the lattice analogue of the $SU(2)$ anomaly first discovered by Witten.

1. Introduction

The formulation of chiral lattice gauge theories on firm theoretical grounds has recently turned out to be an interesting and rapidly developing field [1]. At the root of the recent achievements is the realization that the Nielsen-Ninomiya theorem [2] can be circumvented whenever the lattice Dirac operator fulfils the Ginsparg-Wilson relation [3]

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D \ . \tag{1.1}$$

Lattice actions defined using these kind of operators exhibit good chiral properties [4–6]. In particular, Lüscher has recently proved that within this framework chiral theories with manifest gauge invariance can be defined on the lattice without compromising the theory in any other way [7, 8].

In this work we apply this formalism to the study of global anomalies on the lattice. In the continuum theory global anomalies were first discovered by Witten [9], who proved the mathematical inconsistency of the $SU(2)$ gauge theory coupled to an odd number of doublets of Weyl fermions. Let us briefly sketch his argumentation.

Consider an $SU(2)$ gauge theory coupled to a single doublet of massless Dirac fermions. In terms of the gauge potential A_μ and the fermion fields $\psi, \overline{\psi}$ the action reads

$$S(A_\mu, \psi, \overline{\psi}) = S_g(A_\mu) + \int \mathrm{d}^4x \, \overline{\psi} D\psi. \tag{1.2}$$

S_g denotes the pure gauge part of the action and D stands for the hermitian Dirac operator. In the vacuum sector, after performing the fermion integration, we obtain the following partition function

$$\mathcal{Z}_{\text{Dirac}} = \int \mathcal{D}[A]\mathcal{D}[\psi]\mathcal{D}[\overline{\psi}] \, \mathrm{e}^{-S(A,\psi,\overline{\psi})} = \int \mathcal{D}[A] \, \det D(A) \, \mathrm{e}^{-S_g(A)}. \tag{1.3}$$

In $SU(2)$, the fermion determinant is real. In addition, one doublet of Dirac fermions is equivalent to two doublets of left-handed Weyl fermions. Therefore having only one doublet of Weyl fermions the partition function is given by

$$\mathcal{Z}_{\text{Weyl}} = \int \mathcal{D}[A]\mathcal{D}[\psi]\mathcal{D}[\overline{\psi}] \, \mathrm{e}^{-S(A,\psi,\overline{\psi})} = \int \mathcal{D}[A] \, \sqrt{\det D(A)} \, \mathrm{e}^{-S_g(A)} \ . \tag{1.4}$$

Here the sign of the square root is undetermined. Fixing the sign is therefore part of the complete definition of the chiral gauge theory. Let us arbitrarily fix the sign for the classical vacuum configuration. If we insist on a smooth gauge field dependence of $\sqrt{\det D(A)}$, the sign is also fixed for all gauge fields in the vacuum sector. However, using this prescription for the sign the square root of the Dirac operator is not gauge invariant. Witten has shown this as follows.

One first notices that in four dimensions there exist $SU(2)$ gauge transformations that cannot be continuously deformed to the identity mapping. In mathematical terms this is expressed by saying that the fourth homotopy group of $SU(2)$ is non-trivial: $\pi_4[SU(2)] = \mathbb{Z}_2$.

Now take any gauge transformation $g(x)$ in the non-trivial homotopy class of $SU(2)$ and consider the path

$$A_\mu^t = tA_\mu + (1-t)A_\mu^g \ , \quad t \in [0,1] \ , \tag{1.5}$$

in the vector space of gauge potentials. This path is well defined and interpolates smoothly between A_μ and its gauge transform

$$A_\mu^g = g(A_\mu + \partial_\mu)g^{-1}. \tag{1.6}$$

Using the Atiyah-Singer index theorem Witten proved that an odd number of eigenvalues of the square root of the Dirac operator crosses zero as a function of t. This leads to the switch of sign

$$\sqrt{\det D(A)} = -\sqrt{\det D(A^g)}. \tag{1.7}$$

Hence gauge invariance is violated.

On the other hand we could define the sign always positive, taking the absolute value of the square root of $\det D(A)$. In that way gauge invariance is achieved but the smoothness of $\sqrt{\det D(A)}$ has been sacrificed. The square root of the fermion determinant would have a cusp for the t value in which the crossing of the eigenvalues takes place. In conclusion, Witten's anomaly simply states that the sign of the square root for $\det D(A)$ cannot be defined in a smooth and gauge invariant way.

As has become clear, global anomalies are related to the properties of the space of gauge orbits. In perturbation theory the gauge fixing procedure makes it impossible that the gauge field goes along full orbits and prevents the observation of global obstructions. This is the reason why global anomalies are referred to as non-perturbative.

The formulation of chiral lattice gauge theories recently proposed by Lüscher, besides being non-perturbative, is manifestly gauge invariant. As a consistency requirement of the whole approach, both perturbative and non-perturbative continuum anomalies should cancel in the lattice formulation. Concerning the perturbative ones the status is satisfactory from a theoretical point of view. Only those theories that are perturbatively anomaly free in the continuum can be consistently formulated on the lattice [8]. Now it is natural to ask whether the known non-perturbative anomalies are also reproduced by the lattice theory. The aim of this work is to clarify this issue.

The paper is organized in the following way. We first define Weyl fermions on the lattice and discuss the gauge field dependence of the corresponding projection operators (section 2), a new feature due to the Ginsparg-Wilson relation. In section 3, following reference [8], we review the definition of a fermion measure for chiral lattice gauge theories and define what is meant by a global anomaly. In section 4 we investigate the $SU(2)$ gauge theory with a single doublet of Weyl fermions. We will find that this theory shows a global anomaly in the sense previously defined. The anomaly for higher $SU(2)$ representations is discussed in section 5. Finally we show that this anomaly is equivalent to the Witten anomaly in the continuum (section 6). The last section is devoted to conclusions and the most technical derivations are deferred to Appendix A.

This work is intimately connected to ref. [8] to which the reader is referred for full explanations on some results used here. We also carry over completely the notations used there.

2. *Weyl fermions on the lattice*

2.1 Preliminaries

We consider space-time to be restricted to the sites of a finite euclidean lattice with periodic boundary conditions. Gauge fields are specified by group elements on the bonds joining consecutive sites. Fermionic fields are represented by variables defined on the lattice sites. In the following we consider that the gauge field couples to a multiplet of left-handed fermions given in some unitary representation R of the gauge group.

As has been shown recently [7, 8] a proper definition of these lattice theories is possible if the Dirac operator satisfies the Ginsparg-Wilson relation. An explicit example has been proposed by Neuberger [10]. However, the locality and differentiability with respect to the gauge field of this operator is only guaranteed if the gauge field satisfies the bound

$$\|1 - R[U(p)]\| \le \frac{1}{30} \tag{2.1}$$

for all plaquettes $U(p)$ [11]. Fields compatible with this bound are called *admissible* and we restrict ourselves to those.

The Ginsparg-Wilson relation leads to exact chiral symmetry on the lattice [12]. The fermion action splits into a left- and a right-handed part if we introduce the chiral projectors [13, 14]

$$\hat{P}_{\pm} = \frac{1}{2}[1 \pm \gamma_5(1 - aD)] \ , \quad P_{\pm} = \frac{1}{2}(1 \pm \gamma_5) \ . \tag{2.2}$$

In particular, left-handed fields are defined by the the conditions

$$\hat{P}_{-}\psi = \psi \ , \tag{2.3}$$
$$\overline{\psi} P_{+} = \overline{\psi} \ . \tag{2.4}$$

The fermionic part of the lattice action describing left-handed (Weyl) fermions only therefore reads

$$S_{F,L} = a^4 \sum_x \overline{\psi}(x)[P_+ D \hat{P}_- \psi](x) \, . \tag{2.5}$$

Note that $\hat{P}_-$ contains the Dirac operator D. The definition of left-handed fields therefore depends on the particular background gauge field. This has some geometrical implications which will be relevant later for our investigation of global anomalies.

2.2 Paths in configuration space

We will be interested in trajectories in configuration space. For $t \in [0, 1]$ a trajectory or path $\Gamma(t)$ is defined by specifying the configuration of the gauge field, $U_t(x, \mu)$, at each value of the parameter t.

We require the gauge field to be a smooth function of t. Along paths in configuration space we will use the notation $P_t \equiv \hat{P}_-|_{U=U_t}$ to refer to $\hat{P}_-$ along the path. Eq. (2.3) then implies that the subspace of left-handed fields does not remain constant but rather rotates according to the evolution of P_t. We denote by Q_t the operator transporting P_t along the path. This operator is defined to be the solution of the differential equation

$$\partial_t Q_t = [\partial_t P_t, P_t] Q_t \ , \qquad Q_0 = \mathbb{1} \, . \tag{2.6}$$

The hermiticity of P_t implies that Q_t is an unitary operator. That Q_t is indeed the transporter of P_t can be seen by verifying the relation

$$P_t Q_t = Q_t P_0 . \tag{2.7}$$

Therefore if ψ_0 is a left-handed field at the point $t = 0$, at a different point t we have $P_t \, Q_t \, \psi_0 \ = \ Q_t \, \psi_0$. This means that $Q_t \, \psi_0$ is a left-handed field at the point t.

The definition of Q_t implicitly assumes that the gauge field $U_t(x, \mu)$ is differentiable for all t. However, it can be extended straightforwardly to piecewise smooth paths. For that one defines Q_t^k using eq. (2.6) for each individual smooth piece $\Gamma^k(t)$. The operator Q_t along the whole path is then the ordered product

$$Q_t = \prod_{k=1}^{n} Q_t^k \ . \tag{2.8}$$

This definition is consistent, that is whenever we split a path in more pieces than necessary we will obtain the same result for Q_t. The unitarity of Q_t follows from the unitarity of the individual Q_t^k.

2.3 Definition of the twist

Consider now closed loops in configuration space, that is $U_0(x, \mu) = U_1(x, \mu)$. After a whole turn the subspace of left-handed fields has rotated following the gauge field geometry along the loop. A priori there is no reason to expect the operator Q_1 to be the identity mapping.

As a measure for this rotation we introduce the *twist* $\mathcal{T}$ for any closed loop Γ in configuration space, defined by

$$\mathcal{T}(\Gamma) = \det\left[1 - P_0 + P_0\, Q_1\right] \,. \tag{2.9}$$

Note that the definition of the determinant includes the operator $(1 - P_0)$, representing the identity in the space of right-handed fields at $t = 0$. This operator ensures a well defined determinant in the whole space of fermion fields.

The unitarity of Q_t implies that $\mathcal{T}$ is phase for all gauge groups. If Q_1 is the identity, we get trivially a twist equal to one. The twist for two closed loops with the same base point satisfies the composition law

$$\mathcal{T}(\Gamma_1 \circ \Gamma_2) = \mathcal{T}(\Gamma_1) \cdot \mathcal{T}(\Gamma_2) \,. \tag{2.10}$$

In the following the twist is called non-trivial when it is different from one. Closed loops with non-trivial twist will play an important role for the discussion of global anomalies throughout the next sections.

3. Fermion integration measure and global anomalies

3.1 Definition of the fermion measure

To set up a *quantum* field theory we need to define a functional integral in order to compute correlation functions. In the present approach to chiral gauge theories the projector $\hat{P}_-$ depends on the gauge field [cf. eq. (2.3)]. This implies that the fermion measure for left-handed fields is also gauge-field-dependent. Choosing a measure valid for all gauge field configurations is therefore a non-trivial task. On the other hand, the cancellation of the gauge anomaly and the locality of the resulting theory are fundamental properties that one needs to preserve when defining the fermion integration measure.

In [8] it is shown that the definition of the fermion measure is equivalent to finding a Lie algebra valued current $j_\mu(x) = j_\mu^c(x)\, T^c$, where c denotes the colour index, satisfying various conditions:

1. Smoothness condition: $j_\mu(x)$ has to be a smooth function of the gauge field.

2. Locality condition: $j_\mu(x)$ has to be a local expression in the gauge field.

3. Gauge invariance condition: The current is gauge covariant and satisfies the anomalous conservation law

$$(\nabla_\mu^* j_\mu)^c(x) = \mathcal{A}^c(x), \tag{3.1}$$

$$\mathcal{A}^c(x) = \frac{ia}{2} \mathrm{tr}\left\{\gamma_5 R(T^c) D(x,x)\right\} \,, \tag{3.2}$$

where $R(T^c)$ denotes the generator of the gauge group, ∇_μ^* is the gauge covariant backward lattice derivative and $D(x,x)$ is the kernel of the Dirac operator. The expression $\mathcal{A}^c(x)$ represents the covariant anomaly on the lattice.

The formulation of the fourth and last condition requires some definitions. For any smooth path $U_t(x,\mu)$, $0 \le t \le 1$, we define the *Wilson line*

$$W = \exp\left\{ i \int_0^1 \mathrm{d}t \mathfrak{L}_\eta \right\}, \tag{3.3}$$

where the so-called *measure term* $\mathfrak{L}_\eta$ is defined by

$$\mathfrak{L}_\eta \;=\; a^4 \sum_x \eta_\mu^c(x) j_\mu^c(x), \qquad a\eta_\mu(x) = \partial_t U_t(x,\mu) U_t(x,\mu)^{-1}. \tag{3.4}$$

The Wilson line depends on the chosen current $j_\mu^c(x)$ and will be the total change of phase of the reconstructed fermion measure along the path.

4. Integrability condition: $j_\mu^c(x)$ has to satisfy

$$W = \mathcal{T} \tag{3.5}$$

for all closed curves in configuration space. The twist $\mathcal{T}$ is defined in (2.9).

Whenever a current meets all four conditions, a fermion measure can be consistently defined. The chiral lattice gauge theory is then – up to a constant phase in the path integral – fully specified. This result sometimes goes under the name *Reconstruction Theorem* and the details can be found in [8]. We want to emphasize that neither the locality of the theory is violated nor is gauge invariance broken within this framework.

The first condition guarantees that the reconstructed fermion measure depends smoothly on the gauge field. The locality of the entire theory is preserved as long as condition 2 holds. Once the third condition is fulfilled, the fermion measure is gauge invariant for infinitesimal gauge transformations. Finally, the measure is path-independent if and only if the integrability condition holds.

We remark that the conditions are only sensitive to the axial vector part of the current. Once we have found a current compatible with the conditions, we can easily project out the axial vector part of it. This will also satisfy the conditions. Hence without loss of generality we ask for a current that transforms as an axial vector under the lattice symmetries.

3.2 Global anomalies

It is by no means clear that for a given gauge group and a fixed lattice spacing a an appropriate current $j_\mu(x)$ exists which satisfies all four conditions. Even if we know the existence it is fairly difficult to construct it. So far the existence of a current has been rigorously proved only for the abelian gauge group $U(1)$ [7]. The proof for the non-abelian case is still missing.

In the following we speak of a *global anomaly* if any current satisfying the first three conditions necessarily violates the last one. In that case no proper fermion measure exists. Such an anomaly is therefore an insurmountable obstruction in formulating a chiral lattice gauge theory. The reason why we call this a global anomaly will become clear later.

In a semi-classical analysis global anomalies arise in the following way. Consider classical fields, originating from some smooth gauge potential $A_\mu(x)$ by the path-ordered exponential

$$U(x,\mu) = \mathcal{P}\exp\left\{\int_0^1 \mathrm{d}t A_\mu(x+(1-t)a\hat{\mu})\right\}. \tag{3.6}$$

If the current j_μ^c is a local and smooth function of the gauge field, as required by the first two conditions, it can be expanded in powers of the lattice spacing a. Moreover, if the current satisfies the gauge invariance condition, j_μ^c is a local gauge covariant polynomial of dimension 3 in the gauge potential that transforms as an axial vector. Therefore the expansion in the lattice spacing starts linearly in a, i.e.

$$j_\mu^c(x) = 0 + \mathcal{O}(a). \tag{3.7}$$

So far we employed the first three conditions only. Any current compatible with them vanishes in the classical continuum limit. This immediately implies

$$W = 1 + \mathcal{O}(a) \tag{3.8}$$

for the Wilson line (3.3) along a smooth curve of classical fields. This can lead to a conflict with the integrability condition (3.5) if the twist along such a closed curve is non-trivial in the continuum limit.

In that case, any current satisfying the first three conditions will necessarily violate the integrability condition for some small enough lattice spacing a. Consequently, a proper fermion measure cannot be defined.

4. *Global anomaly in* SU(2)

We now consider the special case when the gauge field is in the fundamental representation ($j = 1/2$) of $SU(2)$. It will be shown that there exist loops in configuration space with a non-trivial twist. Higher $SU(2)$ representations are discussed in section 5.

4.1 Definition of closed loops

Due to the reality properties of $SU(2)$ the twist is real and therefore equal to ± 1. Since these values cannot change continously the twist is a homotopy invariant for $SU(2)$. Concerning global anomalies, violations of the integrability condition will manifest themselves as closed loops in configuration space with a twist equal to -1. In the following we will explicitly construct one such loop.

To begin with we take a homotopically non-trivial continuum gauge transformation $g(x)$. The particular form of $g(x)$ is not needed. Restricting the space-time points to the sites of an euclidean lattice defines a lattice gauge transformation. For $t \in [0, 1]$ we define two paths in configuration space

$$\Gamma_1 : \qquad U_t(x, \mu) = g(x)^t g(x + a\mu)^{-t} \tag{4.1}$$

$$\Gamma_2 : \qquad U_t(x, \mu) = [g(x)g(x + a\mu)^{-1}]^{1-t} \tag{4.2}$$

On the lattice all gauge transformations are smoothly connected with the identity mapping. In particular, Γ_1 is a smooth curve made up of $SU(2)$ gauge transformations. This path is a pure lattice artifact and has no analogue in the continuum. Γ_2 also connects the vacuum configuration with its gauge transformed and is just the lattice version of the path (1.5).

Consider the closed loop $\Gamma_2 \circ \Gamma_1$. It starts and ends at the classical vacuum passing through the pure gauge configuration $g(x)g(x+a\mu)^{-1}$. Our aim is computing $\mathcal{T}(\Gamma_2 \circ \Gamma_1)$ in the classical continuum limit.

As an auxiliary tool for the calculation we define a third path. First note that $SU(2)$ can be embedded in a group with a trivial fourth homotopy group π_4, $SU(3)$ say. The vacuum configuration and its gauge transformed by $g(x)$ are connected by a smooth path of gauge transformations in $SU(3)$. To be more precise there exists a path $\Omega(s, x)$, $0 \leq s \leq 1$, with boundary values [16]

$$\Omega(0, x) \;=\; \mathbb{1}\,, \qquad \Omega(1, x) \;=\; \left(\begin{array}{c|c} g(x) & \\ \hline & 1 \end{array}\right) . \tag{4.3}$$

Here the fundamental representation of $SU(3)$ has been considered. Having this Ω at hand we define a third path on the lattice by

$$\Gamma_3 : \quad U_s(x, \mu) = \Omega(s, x)\Omega(s, x + a\mu)^{-1}, \tag{4.4}$$

which connects the vacuum with the pure gauge configuration $g(x)g(x + a\mu)^{-1}$ in the $1/2 \oplus 0$ representation of $SU(2)$. If we enlarge the first two paths also to this representation we can use the composition law (2.10),

$$\mathcal{T}(\Gamma_2 \circ \Gamma_1) \;=\; \mathcal{T}(\Gamma_2 \circ \Gamma_3) \cdot \mathcal{T}(-\Gamma_3 \circ \Gamma_1), \tag{4.5}$$

to compute the twist along $\Gamma_2 \circ \Gamma_1$. The additional singlet representation does not affect our final result. To see this one first notices the relation

$$\mathcal{T}(\Gamma_{j_1 \oplus j_2}) \;=\; \mathcal{T}(\Gamma_{j_1}) \cdot \mathcal{T}(\Gamma_{j_2}), \tag{4.6}$$

that holds for a direct sum of two $SU(2)$ representations. In addition the twist is equal to 1 for the singlet representation. Therefore the left-hand side of (4.5) is also equal to the twist in the fundamental representation alone.

Two comments should be made. We want to emphasize that the path Γ_2 is well-defined in the space of admissible fields provided that the lattice spacing a is small enough. In the following we always assume this to be the case. The fields along Γ_1 and Γ_3 trivially satisfy the bound (2.1) because they are pure gauge transformations.

The paths Γ_i, $i = 1, 2, 3$, are smooth. However, the loops we have defined out of them are only piecewise smooth. At the contact points of the paths in the vacuum and the pure gauge configuration $g(x)g(x + a\mu)^{-1}$ the loops are not differentiable with respect to t.

4.2 Twist along closed gauge loops

A closed expression for the twist can be derived for pure gauge loops. The discussion of this section applies to any gauge group, only in the end we will restrict ourselves to our particular gauge loop in $SU(3)$.

Given an initial gauge field $U_0(x, \mu)$ gauge paths in configuration space are of the form

$$U_t(x, \mu) = \Lambda_t(x)\, U_0(x, \mu)\, \Lambda_t(x + a\mu)^{-1} \ , \tag{4.7}$$

where Λ_t denotes a curve of gauge transformations with initial value $\Lambda_0 = \mathbb{1}$. The projector to the subspace of left-handed fermion fields along gauge paths is thus given by

$$P_t = \Lambda_t P_0 \Lambda_t^{-1} \ . \tag{4.8}$$

For gauge paths the differential equation (2.6) for the evolution operator Q_t can be solved. In terms of $X_t = \Lambda_t^{-1}\partial_t\Lambda_t$ the solution reads

$$Q_t = \Lambda_t\, \mathcal{P} \exp\left(- \int_0^t \mathrm{d}s\, \mathcal{Y}_s \right) \ , \tag{4.9}$$

where we defined $\mathcal{Y}_s = \{P_0 X_s P_0 + (1 - P_0)X_s(1 - P_0)\}$. Evidently this operator is block diagonal in the space of chiral fields: $\mathcal{Y}_s = \mathcal{Y}_{s,L} + \mathcal{Y}_{s,R}$. Taking this into account for closed gauge loops ($\Lambda_1 = \mathbb{1}$) the determinant defining the twist can be expressed as

$$\mathcal{T} = \exp\left(-ia^4 \sum_x \mathcal{A}^c_{t=0}(x) \int_0^1 \mathrm{d}t\, X^c_t(x) \right) \ . \tag{4.10}$$

That is, along closed gauge loops the twist is proportional to the exponential of the covariant anomaly for the starting configuration.

Our derivation of this result assumes the gauge loop to be smooth. This is not the case for $\Gamma_1 \circ -\Gamma_3$ we are interested in. However, eq. (4.9) can be applied independently to Γ_1 and Γ_3. Taking the product of the two solutions for Q we arrive again at (4.10).

Now we notice that $\Gamma_1 \circ -\Gamma_3$ contains the vacuum configuration. There the anomaly is a translational invariant field, i.e. $\mathcal{A}^c(x) = \mathcal{A}^c(-x)$.[1] Moreover, it is a pseudo-scalar field and therefore odd under parity: $\mathcal{A}^c(x) = -\mathcal{A}^c(-x)$. This implies that the anomaly vanishes in the vacuum of $SU(N)$. An immediate consequence is the result

$$\mathcal{T}(\Gamma_1 \circ -\Gamma_3) = 1 \, . \tag{4.11}$$

[1] It is only here that the periodic boundary conditions of our lattice are used.

Along our closed gauge loop the twist is trivial.

4.3 The twist along the non-gauge loop $\mathcal{T}(\Gamma_2 \circ \Gamma_3)$

We are now left with the calculation of the twist for the loop $\Gamma_2 \circ \Gamma_3$. This presents the added difficulty of containing a non-gauge part, the path Γ_2. Instead of solving the differential equation for Q_t we will derive an integral representation for the twist which we will use for the computation of $\mathcal{T}(\Gamma_2 \circ \Gamma_3)$.

So far we considered loops in configuration space, defined by some gauge field U_t, depending on the parameter t. In the following we assume the gauge field to be smoothly dependent on two parameters, t and s, both lying in the range $[0, 1]$. In addition, the field $U_{t,s}$ should have the following two properties:

$$1)\ U_{0,s} = U_{1,s} = \mathbb{1} \quad 0 \le s \le 1\,, \tag{4.12}$$

$$2)\ U_{t,0} \equiv \mathbb{1}\,, \quad 0 \le t \le 1\,. \tag{4.13}$$

The first one tells us that t parameterizes closed loops with constant base point $\mathbb{1}$ for all allowed values of s. In the following we will simply write $\Gamma(s)$ instead of $\Gamma(U_{t,s})$. One can think of s as an deformation parameter defining a homotopy between the two loops $\Gamma(0)$ and $\Gamma(1)$. The second property means that $\Gamma(0)$ is a constant loop with a single configuration, the classical vacuum.

Now that the gauge field depends on two parameters, the same is true for the left-handed projector, $\hat{P}_- = P_{t,s}$, and the evolution operator $Q = Q_{t,s}$. Consider the twist for the loop $\Gamma(s)$. Starting from the definition of the twist one can easily verify the relation

$$\partial_s \ln \mathcal{T}(\Gamma(s)) = \mathrm{Tr}\,(P_{0,s}\, Q_{1,s}^{-1}\, \partial_s Q_{1,s})\ . \tag{4.14}$$

Since $P_{0,s} = P_{1,s}$ the projector $P_{0,s}$ commutes with $Q_{1,s}$. For the derivative of $Q_{t,s}$ with respect to s an integral representation can be found. Let us make the ansatz $\partial_s Q_{t,s} = Q_{t,s} R_{t,s}$. If we differentiate this equation with respect to t and the differential equation (2.6) with respect to s, we find

$$\partial_s Q_{t,s} = Q_{t,s} \int_0^t \mathrm{d}r\; Q_{r,s}^{-1} \partial_s X_{r,s}\, Q_{r,s}\ . \tag{4.15}$$

Here we defined $X_{r,s} = [\partial_r P_{r,s}, P_{r,s}]$. Setting t equal to 1 eq. (4.14) can alternatively be written as

$$\partial_s \ln \mathcal{T}(\Gamma(s)) = \int_0^1 \mathrm{d}t\ \mathrm{Tr}\, P_{t,s}\, [\partial_t P_{t,s}, \partial_s P_{t,s}]\ . \tag{4.16}$$

If we integrate this equation over s we finally find an integral representation for the twist along $\Gamma(1)$,

$$\mathcal{T}(\Gamma(1)) = \exp \int_0^1 \int_0^1 \mathrm{d}t\, \mathrm{d}s\ \mathrm{Tr}\, P_{t,s}\, [\partial_t P_{t,s}, \partial_s P_{t,s}]\ . \tag{4.17}$$

It is only here that (4.13) enters the result in terms of $\mathcal{T}(\Gamma(0)) = 1$. One should keep in mind that the integral representation (4.17) is valid provided the loop $\Gamma(1)$ can be smoothly contracted to $\Gamma(0)$. This might not be the case for all possible loops in the space of admissible gauge fields. The bound (2.1) is responsible for a highly non-trivial topology of that space.

Formula (4.17) looks not very accessible. In fact, the twist along $\Gamma(1)$ is expressed as an integral over the surface defined by $U_{t,s}$. Notice that the right-hand side is independent of the particular parameterization of the surface. However, the trace in (4.17) can be expanded in powers of the lattice spacing a. This is sufficient to compute the twist in the classical continuum limit.

Consider $U_{t,s}$ to be a homotopy of classical fields. According to (3.6) the link field $U_{t,s}$ is given as the path-ordered exponential of the gauge potential $A_\mu(t, s, x)$, depending also on the parameters t and s. The expansion of the trace was already discussed in ref. [8]. There the leading term is found to be

$$\mathrm{Tr}\, P_{t,s}[\partial_t P_{t,s}, \partial_s P_{t,s}] = -ic_2 \int \mathrm{d}^4x\, d^{abc} \epsilon_{\mu\nu\rho\sigma} \partial_t A_\mu^a(t, s, x) \partial_s A_\nu^b(t, s, x) F_{\rho\sigma}^c(t, s, x)\ , \tag{4.18}$$

where $F^c_{\rho\sigma}(t,s,x)$ denotes the field tensor associated with the gauge potential. The constant c_2 equals $1/32\pi^2$ and d^{abc} is the completely symmetric d-symbol of the gauge group, defined by

$$d^{abc} = 2i\,\mathrm{tr}\{T^a[T^bT^c + T^cT^b]\}\ . \tag{4.19}$$

If we plug expansion (4.18) into (4.17) we are in principle able to compute the twist in the classical continuum limit.

Let us now turn to the loop $\Gamma_2 \circ \Gamma_3$ we are interested in. All we need is a parameterization of the surface enclosed by it. Even less we only need the parameterization in the classical continuum limit. A simple example is given by

$$A_\mu(t,s,x) = (1-t)\Omega(s,x)\partial_\mu\Omega(s,x)^{-1}\,, \tag{4.20}$$

with $\Omega(s,x)$ as introduced before. This surface has the correct boundary. For t equal to 0 we recover the correct continuum limit of Γ_3, defined in (4.4). On the other hand, setting s equal to 1, eq. (4.20) coincides with (1.5) for the classical vacuum configuration, the continuum limit of Γ_2. Note that the parameters t and s do not coincide with the ones given in (4.12) and (4.13). We instead made use of the parametrization invariance of the integral in (4.17) to define a surface matching our particular boundary paths in the classical continuum limit.

The surface parameterization leads directly to the following relations, needed in (4.18).

$$\begin{aligned}
\partial_s A_\mu(t,s,x) &= (1-t)\left\{[\Omega\partial_\mu\Omega^{-1}, \Omega\partial_s\Omega^{-1}] + \partial_\mu(\Omega\partial_s\Omega^{-1})\right\} \\
\partial_t A_\nu(t,s,x) &= -\Omega\partial_s\Omega^{-1} \\
F_{\rho\sigma}(t,s,x) &= (t^2-t)\,[\Omega\partial_\rho\Omega^{-1}, \Omega\partial_\sigma\Omega^{-1}]
\end{aligned} \tag{4.21}$$

For brevity we have suppressed the dependence on (s,x) in Ω. The integration over t in (4.17) is easily performed and gives a factor 1/12. Making use of the anti-symmetry property of the ϵ-tensor we finally find

$$\ln\mathcal{T}(\Gamma_2\circ\Gamma_3) = \frac{-1}{48\pi^2}\int_0^1 \mathrm{d}s \int \mathrm{d}^4x\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\left(\Omega\partial_s\Omega^{-1}\Omega\partial_\mu\Omega^{-1}\cdots\Omega\partial_\sigma\Omega^{-1}\right) \tag{4.22}$$

for the leading term in the expansion in a. Note that we multiplied back the derivative of the potential and the field tensor into the trace over the group generators. Keep also in mind that they generate the fundamental representation of $SU(3)$.

To bring the integral into a more familiar form we define $x_4 = s$ and introduce a five dimensional tensor $\epsilon_{\mu\nu\rho\sigma\lambda}$ with $\epsilon_{01234} = 1$. The logarithm of the twist can thus be written as

$$\ln\mathcal{T}(\Gamma_2\circ\Gamma_3) = \frac{-1}{240\pi^2}\int_{0\le x_4\le 1} \mathrm{d}^5x\,\epsilon_{\mu\nu\rho\sigma\lambda}\,\mathrm{tr}(\Omega\partial_\mu\Omega^{-1}\cdots\Omega\partial_\lambda\Omega^{-1}). \tag{4.23}$$

A factor 1/5 compensates the extra terms due to the fifth index of the anti-symmetric tensor.

This type of integrals is well known in the context of anomalies in the continuum. Usually one considers mappings $\Omega(s,x)$ with boundary values $\Omega(0,x) = \Omega(1,x) = \mathbb{1}$. In that case the integral is an integer multiple of $2\pi i$. In (4.23), however, the boundary values are given by (4.3). Even in this case the possible values are strongly restricted. First note that the integral is invariant under small variations

$$\delta\Omega(s,x) \;=\; \omega^c(s,x)T^c\,, \tag{4.24}$$

as long as the variation preserves the particular form of the boundary values (4.3). Therefore, the integral depends only on the homotopy class of $g(x)$. If $g(x)$ is in the non-trivial homotopy class, $g^2(x)$ belongs to the trivial one. This already implies that the integral in (4.23) is either 0 or $i\pi$ (mod $2\pi i$).

An explicit computation, performed by Witten in [15], yielded the value $i\pi$. Hence our final result is

$$\mathcal{T}(\Gamma_2 \circ \Gamma_3) = -1\,, \tag{4.25}$$

in the classical continuum limit.

4.4 Global anomaly in SU(2)

Taking into account result (4.11) the composition law (2.10) reads

$$\mathcal{T}(\Gamma_2 \circ \Gamma_1) = \mathcal{T}(\Gamma_2 \circ \Gamma_3). \tag{4.26}$$

As we have already mentioned the left hand side of this equation can only take the values ± 1. Since (4.26) is valid for all lattice spacings, it implies that (4.25) is not only the result in the continuum limit, but it is also the twist for a sufficiently small lattice spacing a. The absence of terms proportional to a in (4.25) has its reason in our special loop which starts in the vacuum configuration. So our final result reads

$$\mathcal{T}(\Gamma_2 \circ \Gamma_1) = -1. \tag{4.27}$$

We want to emphasize that the introduction of Γ_3 is merely a technical tool. If we were able to compute directly the evolution operator Q_t along $\Gamma_2 \circ \Gamma_1$, we would have found the same twist using directly its definition (2.9).

We made use of the fact that $\Gamma_2 \circ \Gamma_1$ starts in the classical vacuum configuration. Our result (4.27), however, is not restricted to such loops. As we have previously remarked, the twist is a homotopy invariant in $SU(2)$. Having found a loop with a twist equal to -1, all loops in the same homotopy class have the same twist. This holds true even though the loop we have considered is only piecewise smooth, since it can be deformed to a totally smooth one.

To establish a global anomaly we have to show $W \neq \mathcal{T}$ along $\Gamma_2 \circ \Gamma_1$. The semi-classical argument in section 3.2 cannot directly be applied to this loop, because Γ_1 has no continuum limit. However, the current is of order a along Γ_2. The path Γ_1 is a curve of gauge transformations and because the current is gauge-covariant we can conclude that the current is of order a along the entire loop. Hence for the Wilson line along $\Gamma_2 \circ \Gamma_1$ we find

$$W(\Gamma_2 \circ \Gamma_1) \;=\; 1 + \mathcal{O}(a). \tag{4.28}$$

According to our discussion in section 3, the loop $\Gamma_2 \circ \Gamma_1$ thus violates the integrability condition and an $SU(2)$ gauge theory exhibits a global anomaly.

As we have already pointed out, the integral in (4.23) was already encountered in the context of the Witten anomaly. In ref. [16] the Witten anomaly is calculated as the cumulative effect of the perturbative non-abelian anomaly along a path representing the continuum version of Γ_3. The change in the phase of the effective action under homotopically non-trivial $SU(2)$ gauge transformations is precisely given by the integral in (4.23). This already indicates a connection between Witten's anomaly and the obstruction we found. In section 6 we will have a closer look at this.[2]

5. Global SU(2) anomaly in higher representations

5.1 Preliminaries

So far we considered the fundamental representation of $SU(2)$ only. For computational reasons we introduced an auxiliary path Γ_3 with the gauge field in the fundamental representation of $SU(3)$ and embedded the links of the other paths Γ_1, Γ_2, into this representation.

[2]The reader mainly interested in the connection to Witten's anomaly can safely skip the next section and may continue directly with section 6 in a first reading.

In the following we compute the twist for higher representations of $SU(2)$ along the same lines. However, the embedding turns out to be less trivial. Basically, for any given $SU(2)$ representation j we cannot find an appropriate $SU(3)$ representation R that will decompose into j and a finite number of singlets. More generally, we always end up with a decomposition

$$R \quad \rightarrow \quad \sum_j c(R,j)\, j\,, \tag{5.1}$$

where the coefficients $c(R,j)$ give the multiplicity of the representation j. To turn (5.1) the other way around, only particular sums of $SU(2)$ representations can be embedded in an $SU(3)$ representation R, such that a path of gauge transformations (4.3) exists.

Having this in mind we start with the definition of Γ_3. In analogy to (4.4) we define $\Gamma_3(R)$ with a curve $\Omega_R(s,x)$, this time in the representation R of $SU(3)$. The boundary value $\Omega_R(1,x)$ is still in $SU(2)$, but now in the representation given by the right hand side of (5.1). Consequently we define the paths Γ_1, Γ_2 for this direct sum of $SU(2)$ representations.

Using (4.6) several times we find the generalization of (4.5) to be

$$\prod_j \mathcal{T}(j)^{c(R,j)} \quad = \quad \mathcal{T}(R)\,. \tag{5.2}$$

Notice that $\mathcal{T}(R)$ denotes the twist along $\Gamma_2 \circ \Gamma_3$ and R indicates the representation used for Γ_3. On the other hand, $\mathcal{T}(j)$ stands for the twist along the pure $SU(2)$ loop $\Gamma_2 \circ \Gamma_1$. In (5.2) we already used the fact that the twist along the pure gauge loop $-\Gamma_3 \circ \Gamma_1$ is 1 for all representations R. Relation (5.2) is our master formula that will enable us to compute recursively $\mathcal{T}(j)$ for all $SU(2)$ representations.

5.2 The twist $\mathcal{T}(R)$

Let us discuss the twist $\mathcal{T}(R)$ along $\Gamma_2 \circ \Gamma_3$ for arbitrary R. The integral in (4.23) contains the product of five generators of $SU(3)$. Expressing twice the product of two of them as a commutator, we can write our result (4.23) as

$$\ln \mathcal{T} \quad = \quad d^{abc} I^{abc}, \tag{5.3}$$

where d^{abc} is the d-symbol of $SU(3)$. Notice that d^{abc} is real because of the anti-hermiticity of the generators. The remaining part I^{abc} is independent of the group representation and the explicit expression of it is not needed in the following. For some other representation R the result (4.23) can be similarly written as

$$\ln \mathcal{T}(R) \quad = \quad d_R^{abc} I^{abc}. \tag{5.4}$$

It differs from (5.3) only in the d-symbol that is now defined with the generators $R(T^a)$ of the representation R. One can show that d_R^{abc}can always be expressed in terms of d^{abc}. Introducing the *anomaly coefficient* $A(R)$ we write

$$d_R^{abc} \quad = \quad A(R)\, d^{abc}\,, \tag{5.5}$$

leading to the relation

$$\mathcal{T}(R) \quad = \quad \mathcal{T}^{A(R)}. \tag{5.6}$$

for the twist along $\Gamma_2 \circ \Gamma_3$ in higher representations. Once the twist $\mathcal{T}$ for the fundamental representation is known, its value for higher representations is completely determined by the anomaly coefficient. The computation of $\mathcal{T}(R)$ is therefore reduced to the purely group theoretic task of calculating $A(R)$.

$$
\begin{array}{rcl}
(\frac{1}{2} \oplus 0)\otimes^2 & = & 2 \cdot 0 \oplus \; 2 \cdot \frac{1}{2} \oplus \; 1 \cdot 1 \\
(\frac{1}{2} \oplus 0)\otimes^3 & = & 4 \cdot 0 \oplus \; 5 \cdot \frac{1}{2} \oplus \; 3 \cdot 1 \oplus \; 1 \cdot \frac{3}{2} \\
(\frac{1}{2} \oplus 0)\otimes^4 & = & 9 \cdot 0 \oplus 12 \cdot \frac{1}{2} \oplus \; 9 \cdot 1 \oplus \; 4 \cdot \frac{3}{2} \oplus 1 \cdot 2 \\
(\frac{1}{2} \oplus 0)\otimes^5 & = & 21 \cdot 0 \oplus 30 \cdot \frac{1}{2} \oplus 25 \cdot 1 \oplus 14 \cdot \frac{3}{2} \oplus 5 \cdot 2 \oplus 1 \cdot \frac{5}{2}
\end{array}
$$

Table 1: The decomposition of $(\frac{1}{2} \oplus 0)\otimes^n$ up to n equals 5.

By definition $A(R)$ is equal to one for the fundamental representation. Furthermore one can straightforwardly establish the relations

$$
\begin{array}{rcl}
A(R_1 \oplus R_2) & = & A(R_1) + A(R_2), \\
A(R_1 \otimes R_2) & = & A(R_1)d_2 + A(R_2)d_1,
\end{array}
\tag{5.7}
$$

where d_i denotes the dimension of the representation R_i.

Let us consider a particular example. The fundamental representation of $SU(3)$ is three dimensional and in the following denoted by 3. For the n-fold tensor product $3 \otimes \ldots \otimes 3 = 3\otimes^n$ the anomaly coefficient and the twist are easily computed to be

$$
A(3\otimes^n) \;=\; n3^{n-1}, \qquad \mathcal{T}(3\otimes^n) \;=\; (-1)^n. \tag{5.8}
$$

We will use this result in the next subsection to classify the anomalous $SU(2)$ representations.

5.3 Global SU(2) anomaly in higher representations

As already mentioned, the fundamental representation 3 of $SU(3)$ decomposes into a doublet and a singlet of $SU(2)$. Hence, for the n-fold tensor product $3\otimes^n$ the decomposition (5.1) reads

$$
3\otimes^n \;\;\rightarrow\;\; (\frac{1}{2} \oplus 0)\otimes^n \;\;=\;\; \sum_j c(n,j)\, j\,. \tag{5.9}
$$

Up to n equals 5 we have collected the explicit decompositions in table 1. Consider first $n = 2$. Using (5.2) and (5.8) we find

$$
\mathcal{T}(1) \;\;=\;\; 1\,, \tag{5.10}
$$

because the anomalous doublet representation with $\mathcal{T}(\frac{1}{2}) = -1$ appears twice in the decomposition. On the other hand, $c(3, \frac{1}{2})$ equals 5 and we find

$$
\mathcal{T}(\frac{3}{2})\mathcal{T}(\frac{1}{2}) \;=\; -1 \;\;\Rightarrow\;\; \mathcal{T}(\frac{3}{2}) \;=\; 1\,. \tag{5.11}
$$

Here the minus sign coming from (5.8) is compensated by an additional sign due to an odd number of doublet representations. Compare this with $n = 5$. Here $c(5, \frac{1}{2})$ is even and, because one can check before $\mathcal{T}(2) = 1$, we get

$$
\mathcal{T}(\frac{5}{2}) \;=\; -1\,. \tag{5.12}
$$

These explicit examples should be enough to illustrate the idea of our method. In general we find the following results for all integers n including zero:

$$
\begin{array}{lrcr}
1. & \mathcal{T}(n) & = & 1, \\
2. & \mathcal{T}(2n + \frac{1}{2}) & = & -1, \\
3. & \mathcal{T}(2n + \frac{3}{2}) & = & 1.
\end{array}
\tag{5.13}
$$

The proofs for these statements can be found in appendix A. In summary we find the representations with

$$j = 2n + \frac{1}{2}, \qquad n = 0, 1, 2 \ldots \tag{5.14}$$

to be anomalous.

6. Connection with Witten's anomaly

In the following we will show that the global anomaly we discovered in the previous sections is nothing but the Witten anomaly [9]. Let us consider a smooth closed loop U_t in field space as above. For simplicity we assume D_0 to have no zero modes. The function

$$f(t) = \det\left(1 - P_+ + P_+ D_t Q_t D_0^\dagger\right) \tag{6.1}$$

depends smoothly on t and is real for the gauge group $SU(2)$. In addition it satisfies

$$f(0) > 0, \qquad f(1) = \mathcal{T} f(0). \tag{6.2}$$

This implies that f passes through zero an odd number of times for $0 \le t \le 1$ if and only if $\mathcal{T} = -1$. Next one can show

$$f^2(t) = \det D_t \det D_0^\dagger. \tag{6.3}$$

Because of the γ_5-hermiticity $D^\dagger = \gamma_5 D \gamma_5$, the eigenvalues λ_i of D come in complex conjugate pairs and for the determinant we find

$$\det D_t = \prod_i \lambda_i(t) \lambda_i^*(t). \tag{6.4}$$

The eigenvalues depend smoothly on t for smooth paths. According to (6.3) a passing through zero of $f(t)$ at some point t_0 implies a passing through zero of an odd number of eigenvalues $\lambda_i(t)$. One can prove this by expanding both f and λ_i around t_0 in (6.3). Thus we conclude that an odd number of eigenvalues of D cross zero along a closed loop if and only if $\mathcal{T} = -1$.

We gave an explicit example for such a loop, namely $\Gamma_2 \circ \Gamma_1$. Because Γ_1 is a curve of gauge transformations the crossing of the eigenvalues occurs along Γ_2, the lattice analogue of (1.5). Thus we find the same behaviour of the spectral flow on the lattice that Witten proved in the continuum using the Atiyah–Singer index theorem.

Witten also investigated the anomaly for higher fermion representations. He found the theory to be inconsistent if the representation is such that $2\,\mathrm{tr}\, T_3^2$ is an odd integer. The trace of T_3^2 depends only on j and one finds

$$2\,\mathrm{tr}\, T_3^2 = \frac{2}{3} j(j+1)(2j+1). \tag{6.5}$$

Obviously the right hand side of (6.5) is even if j is an integer, so these representations are anomaly free. For half-integer values one easily proves that $2\,\mathrm{tr}\, T_3^2$ is odd if

$$j = 2n + \frac{1}{2}, \qquad n = 0, 1, 2, \ldots \tag{6.6}$$

As has been shown in the previous section, these are exactly the representations with a twist equal to -1 on the lattice. Notice that this coincidence is highly non-trivial. Equation (6.6) emerges from the investigation of the zero modes of a five dimensional Dirac operator in a certain instanton field. This differs completely from the embedding technique that leads to (5.14).

7. Concluding remarks

In this paper we have addressed for the first time the issue of non-perturbative anomalies beyond the semi-classical level. The lattice formulation provides a mathematically well-defined framework to analyze this problem.

Global anomalies on the lattice are related to a non-trivial twist for closed loops in configuration space. The twist $\mathcal{T}$ has proved to be the essential quantity. In particular we did not consider directly the effective action to establish the anomaly. The twist has the advantage to be also defined in sectors with non-zero topological charge and might be considered to investigate the existence of global anomalies there.

An explicit construction of a non-trivial closed loop in the vacuum sector has been given for the case of an $SU(2)$ gauge theory coupled to a doublet of Weyl fermions. More generally, the anomaly is present whenever we consider representations of $SU(2)$ such that $2\,\mathrm{tr}\,T_3^2$ is an odd integer.

Our treatment of global anomalies has the advantage of being completely analytical and resembles rather the embedding technique used in [16] than the original proof of Witten, based on the spectral flow of the Dirac operator. However, we have shown that Witten's original argument follows immediately from our result.

In Lüscher's approach only those chiral gauge theories can be formulated consistently on the lattice which are perturbatively anomaly free in the continuum [7,8]. This means that a necessary condition for the lattice formulation to exist is an anomaly free fermion multiplet ($d_R^{abc} = 0$). We have seen here that this is also the case for non-perturbative anomalies as far as Witten's anomaly is concerned. However, one needs to exclude other global anomalies. Starting from a fermion representation perturbatively anomaly free it is not obvious that $\mathcal{T}$ will be equal to one for all closed loops. Only in that case one can set $j_\mu^c = 0$. The four conditions listed in section 3 are then satisfied and the chiral gauge theory is completely defined.

We want to thank Martin Lüscher for advise and numerous comments throughout the realization of this work. Thanks go also to Peter Weisz who critically read the first version of this paper.

Appendix A

I. $\mathcal{T}(n) = 1$ for all integers n including zero.

Proof: The statement is true for $n = 0, 1$. Now we assume $\mathcal{T}(k) = 1$ for $k = 1, 2, \ldots, n$. Consider

$$(\frac{1}{2} \oplus 0)\otimes^{2n+2} = (1 \oplus 2 \cdot \frac{1}{2} \oplus 2 \cdot 0)\otimes^{n+1} = 1 \otimes^{n+1} + \sum_j \tilde{c}(2n+2, j) j \,. \tag{A.1}$$

Because both the doublet and the singlet representation appear twice in the decomposition of $(1/2 \oplus 0)\otimes^2$, the coefficients $\tilde{c}(2n+2, j)$ are even for all values of j. The tensor product of the triplet representation can be decomposed further and we write

$$1\otimes^{n+1} = (n+1) \oplus \sum_{k=0}^{n} a_k \, k \,, \tag{A.2}$$

where we introduced the not necessarily even integers a_k. In that decomposition only integer representations occur. Taking into account (5.8) equation (5.2) reads

$$\mathcal{T}(n+1) \prod_k \mathcal{T}(k)^{a_k} \prod_j \mathcal{T}(j)^{\tilde{c}(2n+2,j)} = 1 \,. \tag{A.3}$$

According to our assumption $\mathcal{T}(k) = 1$ this implies $\mathcal{T}(n+1) = 1$. □

II. $\mathcal{T}(2n + \frac{1}{2}) = -1$ and $\mathcal{T}(2n + \frac{3}{2}) = 1$ for all integers n including zero.

Proof: The statement is true for $n = 0$. To prove it for arbitrary integers we first define

$$\Sigma_a(2n + \frac{1}{2}) = \sum_{k=0}^{n-1} c(4n+1, \frac{4k+1}{2}), \qquad n \geq 1 \,, \tag{A.4}$$

$$\Sigma_a(2n + \frac{3}{2}) = \sum_{k=0}^{n} c(4n+3, \frac{4k+1}{2}), \qquad n \geq 0 \,. \tag{A.5}$$

$\Sigma_a(j)$ gives the total number of anomalous representations in the decomposition (5.1) of $(\frac{1}{2} \oplus 0)\otimes^{2j}$ with highest weight less than j. In addition we define

$$\Sigma_f(2n + \frac{1}{2}) = \sum_{k=1}^{n} c(4n+1, \frac{4k-1}{2}), \qquad n \geq 1 \,, \tag{A.6}$$

$$\Sigma_f(2n + \frac{3}{2}) = \sum_{k=1}^{n} c(4n+3, \frac{4k-1}{2}), \qquad n \geq 1 \,. \tag{A.7}$$

$\Sigma_f(j)$ gives the number of anomaly free *half integer* representations in the decomposition of $(\frac{1}{2} \oplus 0)\otimes^{2j}$, again with highest weight less than j. Formulae (5.8) and (5.2) imply

$$\mathcal{T}(j) \cdot (-1)^{\Sigma_a(j)} = -1 \tag{A.8}$$

for half-integer j. In order to prove the statement all we need to show is that $\Sigma_a(2n + \frac{1}{2})$ is even and $\Sigma_a(2n + \frac{3}{2})$ is odd.

Using table (1) one can easily check that $\Sigma_a(5/2)$ is even and $\Sigma_a(3/2)$ odd. Now let us assume that $\Sigma_a(2n + \frac{1}{2})$ is even for some fixed integer $n \geq 1$. We consider the tensor product

$$(\frac{1}{2} \oplus 0)\otimes^{4n+3} = 1 \otimes \left((\frac{1}{2} \oplus 0)\otimes^{4n+1} \right) \oplus \sum_j \tilde{c}(4n+3, j) j \,, \tag{A.9}$$

where all the coefficients $\tilde{c}$ are even integers. The tensor product with the triplet representation can be decomposed into irreducible representations with the result

$$1 \otimes \left((\frac{1}{2} \oplus 0)^{\otimes 4n+1} \right) = \sum_{k=1}^{2n+2} b(4n+3, \frac{2k-1}{2}) \frac{2k-1}{2} \oplus \text{integer representations}\,. \tag{A.10}$$

The coefficients b are given in terms of the coefficients c as

$$\begin{array}{lclclcl} b(4n+3,\frac{1}{2}) & = & c(4n+1,\frac{1}{2}) & + & c(4n+1,\frac{3}{2}), & & \\ \vdots & & \vdots & & \vdots & & \\ b(4n+3,\frac{2k+1}{2}) & = & c(4n+1,\frac{2k-1}{2}) & + & c(4n+1,\frac{2k+1}{2}) & + & c(4n+1,\frac{2k+3}{2}), \\ \vdots & & \vdots & & \vdots & & \vdots \\ b(4n+3,\frac{4n+1}{2}) & = & & & c(4n+1,\frac{4n-3}{2}) & + & c(4n+1,\frac{4n-1}{2}), \\ b(4n+3,\frac{4n+3}{2}) & = & & & & & c(4n+1,\frac{4n-1}{2}). \end{array} \tag{A.11}$$

We denote the contribution of (A.10) to $\Sigma_a(2n+\frac{3}{2})$ by $\tilde{\Sigma}_a(2n+\frac{3}{2})$. Using $c(m,\frac{m}{2})=1$ we find

$$\tilde{\Sigma}_a(2n+\frac{3}{2}) = 1 + \Sigma_a(2n+\frac{1}{2}) + 2\Sigma_f(2n+\frac{1}{2})\,. \tag{A.12}$$

We assumed $\Sigma_a(2n+\frac{1}{2})$ to be even, hence $\tilde{\Sigma}_a(2n+\frac{3}{2})$ is odd. Because the coefficients $\tilde{c}$ in (A.9) are all even, $\Sigma_a(2n+\frac{3}{2})$ is also odd, i.e.

$$\Sigma_a(2n+\frac{1}{2}) \text{ is even} \implies \Sigma_a(2n+\frac{3}{2}) \text{ is odd}\,. \tag{A.13}$$

Having established this result, we consider now the tensor product (A.9) with exponent $4n+5$. Equations (A.9), (A.10) and (A.11) need just be modified by the replacement $n \to n+1/2$. Similar to (A.12) we find

$$\tilde{\Sigma}_a(2n+\frac{5}{2}) = 1 + \Sigma_a(2n+\frac{3}{2}) + 2\,\Sigma_f(2n+\frac{3}{2})\,. \tag{A.14}$$

We therefore conclude:

$$\Sigma_a(2n+\frac{3}{2}) \text{ is odd} \implies \Sigma_a(2(n+1)+\frac{1}{2}) \text{ is even}\,. \tag{A.15}$$

This completes the proof. $\square$

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